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The q-Analogues of Squeezed States and Some Properties[1]

Xiang-guo Meng[2] and Ji-suo Wang[2]

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In this paper we shall introduce two q-analogues of the squeezed states in terms of the technique of integration within an ordered product of operators and the properties of the inverses of q-deformed annihilation and creation operators, and some nonclassical properties of the states are examined. Furthermore, we obtain some new completeness relations composed of the bra and ket which are not mutually Hermitian conjugate.

KEY WORDS: q-analogues of the squeezed state; quadrature squeezing; amplitude-squared squeezing; antibunching effect.

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1. INTRODUCTION

In the past few years there has been paid much attention to q-deformed bosonic oscillator owing to its possible applications in studing the q-analogue of the quantum theory of the radiation light field (Gray and Nelson, 1990; Man'ko *et al.*, 1997). The q-oscillators are interpreted as a nonlinear oscillator with a very specific type of nonlinearity (Man'ko *et al.*, 1993a,b), in which the frequency of vibration depends on the energy of these vibrations through the hyperboblic cosine function containing a nonlinearity patameter. In Kuang and Wang (1993); Wang *et al.* (1995, 1996); Fan (1994); Fan and Jing (1995); Kuang *et al.* (1993); Song and Fan (2002) some important physical notions such as the coherent states, even and odd coherent states and squeezed states for the ordinary oscillators have been extended to the q-deformed case and their nonclassical properties were discussed. In this paper in the same way as in Song and Fan (2002), we shall introduce two normalizable q-analogues of the squeezed states by virtue of the technique

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[2] Department of Physics, Liaocheng University, 252059, Liaocheng, Shandong, P. R. China. e-mail: mengxiangguo1978@sina.com, xgmeng@lcu.edu.cn.

of integration within an ordered product (IWOP) of operators and the properties of the inverses of q-deformed annihilation and creation operators, and some non-classical properties such as quadrature squeezing, amplitude-squared squeezing and antibunching effects etc, of the states are examined.

2. THE Q-ANALOGUES OF SQUEEZED STATES

We first introduce the nonliear annihilation operator a_f and creation operator $a_f^\dagger$ as follows:

$$a_f = a\frac{1}{f(N)}, \quad a_f = \frac{1}{f(N)}a^\dagger \tag{1}$$

where $f(N)$ is an operator-valued function of the number operators $N = a^\dagger a$ and chosen to be real and non-negative, a and $a^\dagger$ are the bosonic annihilation and creation operators, respectively. From Eq. (1) for different nonlinearity function $f(N)$ we can have different annihilation and creation operators a_f and $a_f^\dagger$, in the present paper if we take

$$f(N) = \sqrt{\frac{N}{[N]}}, \quad [N] = (1 - q^N)/(1 - q) \tag{2}$$

then the nonliear annihilation operator a_f and creation operator $a_f^\dagger$ shall respectively become the q-deformed boson annihilation operator a_q and creation operator $a_q^\dagger$, which are distortions of the annihilation operator a and creation operator $a^\dagger$ of the usual harmonic oscillator.

The q-deformed boson creation operator $a_q^\dagger$ and annihilation operator a_q and the number operator $N = a_q^+ a_q$ satisfy the following closed algebraic relations:

$$a_q a_q^\dagger - q a_q^\dagger a_q = 1, \quad [N, a_q^\dagger] = a_q^\dagger, \quad [N, a_q] = -a_q \tag{3}$$

where q is a deformed parameter.

From Song and Zhu (2002), we know that the so-called q-Fock state $|n\rangle_q$ is the usual Fock state $|n\rangle$ in fact, so operating the q-deformed operators a_q, $a_q^\dagger$ and N on the Fock states $|n\rangle$, we have

$$a_q|n\rangle = \sqrt{[n]}|n-1\rangle \tag{4}$$

$$a_q^\dagger|n\rangle = \sqrt{[n+1]}|n+1\rangle \tag{5}$$

$$N|n\rangle = [n]|n\rangle \tag{6}$$

Therefore we can easily obtain the inverses of the operators a_q and $a_q^\dagger$ as follows:

$$a_q^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{[n+1]}} |n+1\rangle\langle n| \tag{7}$$

$$\left(a_q^\dagger\right)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{[n+1]}} |n\rangle\langle n+1| = \left(a_q^{-1}\right)^\dagger \tag{8}$$

It then follows that the inverses of the operators a_q and $a_q^\dagger$ satisfy the following noncommutative relations

$$a_q a_q^{-1} = \left(a_q^\dagger\right)^{-1} a_q^\dagger = 1 \tag{9}$$

$$a_q^{-1} a_q = a_q^\dagger \left(a_q^\dagger\right)^{-1} = 1 - |0\rangle\langle 0| \tag{10}$$

which means a_q^{-1} is the right inverse of a_q and $(a_q^\dagger)^{-1}$ is the left inverse of $a_q^\dagger$. The result is completely analogous to the case of the inverse of ordinary basonic operators.

By virtue of the inverses operators a_q^{-1}, $(a_q^\dagger)^{-1}$ and the number operator N, in the following we construct two new operators:

$$A_q^\dagger = N a_q^{-1}, \quad A_q = \left(a_q^\dagger\right)^{-1} N \tag{11}$$

Using Eqs. (3) and (9–11), we can prove that

$$\left[a_q, A_q^\dagger\right] = a_q N a_q^{-1} - N a_q^{-1} a_q = (N+1) a_q a_q^{-1} - N(1 - |0\rangle\langle 0|) = 1 \tag{12}$$

$$\begin{aligned}\left[A_q, a_q^\dagger\right] &= \left(a_q^\dagger\right)^{-1} N a_q^\dagger - a_q^\dagger \left(a_q^\dagger\right)^{-1} \\ N &= \left(a_q^\dagger\right)^{-1} a_q^\dagger (N+1) - (1 - |0\rangle\langle 0|) N = 1\end{aligned} \tag{13}$$

and

$$\left[A_q^\dagger, N\right] = -A_q^\dagger, \quad [A_q, N] = A_q. \tag{14}$$

Therefore $A_q^\dagger$ $(a_q^\dagger)$ and a_q (A_q) make up fundamental canonical conmmutative relations.

In Fan and Jing (1993), Fan has proved that the normal ordering form of the vaccum projector is

$$|0\rangle\langle 0| =: \exp\left(-a^\dagger a\right): \tag{15}$$

In the same way we can also prove that the normal ordering form of the vaccum projector is

$$|0\rangle\langle 0| = \vdots \exp(-a_q^\dagger A_q) \vdots \tag{16}$$

where the normal ordering $\vdots\,\vdots$ is for the operators $a_q^\dagger$ and A_q, and it completely distincts from the normal ordering form of the vaccum projector $|0\rangle\langle 0|$ in Eq. (15).

Subsequently, we construct the following eigenstates, which the ket and bra are not mutually Hermitian conjugate

$$||x\rangle = \pi^{-1/4}\exp\left[-\frac{1}{2}x^2 + \sqrt{2}xa_q^\dagger - \frac{1}{2}a_q^{\dagger 2}\right]|0\rangle \tag{17}$$

$$\langle x| = \pi^{-1/4}\langle 0|\exp\left[-\frac{1}{2}x^2 + \sqrt{2}xA_q - \frac{1}{2}A_q^2\right] \tag{18}$$

where setting

$$x = \frac{1}{\sqrt{2}}(A_q + a_q^\dagger). \tag{19}$$

By virtue of Eqs. (16)–(19) and the IWOP technique, performing the following integration we can obtain the following completeness relations:

$$\int_{-\infty}^{\infty} dx||x\rangle\langle x| = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dx \vdots \exp\left[-x^2 + \sqrt{2}x\left(a_q^\dagger + A_q\right)\right.$$
$$\left. - \frac{1}{2}\left(a_q^{\dagger 2} + A_q^2\right) - a_q^\dagger A_q\right]\vdots = 1 \tag{20}$$

By means of simple calculation, we can prove that the eigenvalue equations of the coordinate operator x may be written as

$$x||x\rangle = x||x\rangle, \quad \langle x|x = \langle x|x, \tag{21}$$

Then using Eqs. (16)–(19) we have the following squeezing operator

$$S(r) = \int_{-\infty}^{\infty}\frac{dx}{\sqrt{\mu}}||x_1/\mu\rangle\langle x_1|$$
$$= \exp\left(-\frac{1}{2}a_q^{\dagger 2}\tanh r\right)\exp\left[\left(N + \frac{1}{2}\right)\ln \sec\mathrm{hr}\right]\exp\left(\frac{1}{2}A_q^2\tanh r\right) \tag{22}$$

where r is a real squeeze parameter, i.e.,

$$r = e^\mu, \quad \tanh r = (\mu^2 - 1)/(\mu^2 + 1) \tag{23}$$

In terms of the above squeezing operator we can derive the following transformations that the squeezed operator $S\,(r)$ engenders

$$S(r)A_qS^{-1}(r) = A_q \cosh r + a_q^\dagger \sinh r \tag{24}$$

$$S(r)a_q^\dagger S^{-1}(r) = a_q^\dagger \cosh r + A_q \sinh r \tag{25}$$

Operating the squeezed operator $S(r)$ on the vacuum state $|0\rangle$ and the one-photon state $|1\rangle$ respectively, we can obtain the q-analogues of a squeezed vacuum state

$$|r, q\rangle_1 = C_1 \exp\left(-\frac{1}{2} a_q^{\dagger 2} \tanh r\right) |0\rangle$$
$$= C_1 \sum_{n=0}^{\infty} \left(-\frac{1}{2} \tanh r\right)^n \frac{\sqrt{[2n]!}}{n!} |2n\rangle \tag{26}$$

and the q-analogues of a squeezed one-photon state

$$|r, q\rangle_2 = C_2 \exp\left(-\frac{1}{2} a_q^{\dagger 2} \tanh r\right) |1\rangle$$
$$= C_2 \sum_{n=0}^{\infty} \left(-\frac{1}{2} \tanh r\right)^n \frac{\sqrt{[2n+1]!}}{n!} |2n+1\rangle \tag{27}$$

where C_1 and C_2 are the normalization constants of the states $|\lambda, q\rangle_1$ and $|r, q\rangle_2$ and are given by, respectively

$$|C_1|^2 = \left[\sum_{n=0}^{\infty} \left(\frac{1}{2} \tanh r\right)^{2n} \frac{[2n]!}{(n!)^2}\right]^{-1} \tag{28}$$

$$|C_2|^2 = \left[\sum_{n=0}^{\infty} \left(\frac{1}{2} \tanh r\right)^{2n} \frac{[2n+1]!}{(n!)^2}\right]^{-1} \tag{29}$$

$$[n]! = [0][1][2] \cdots [n], \quad [0]! = 1 \tag{30}$$

Here we can define a displacement operator

$$D(z) = \exp\left(z a_q^{\dagger} - z^* A_q\right) \tag{31}$$

and then operate $D(z)$ on the vacuum state $|0\rangle$, we can obtain q-analogues of a coherent state

$$||z\rangle_q = D(z)|0\rangle = \exp\left(-\frac{1}{2}|z|^2 + z a_q^{\dagger}\right) |0\rangle. \tag{32}$$

By virtue of the IWOP technique we can easily prove that the states (32) and the following states

$${}_q\langle z| = \langle 0| \exp\left(-\frac{1}{2}|z|^2 + z^* A_q\right) \tag{33}$$

make up the overcompleteness relations in the following form:

$$\int \frac{dz}{\pi} ||z\rangle_{qq}\langle z| = \int \frac{dz}{\pi} D(z)|0\rangle\langle 0|D^{-1}(z)$$
$$= \int \frac{dz}{\pi} \vdots \exp\left(-|z|^2 + za_q^\dagger + z^* A_q - a_q^\dagger A_q\right)\vdots = 1 \quad (34)$$

On the other hand if operating $D(z)$ on the squeezed vacuum states $|r, q\rangle_1$, we can obtain q-analogues of a squeezed coherent state as follows:

$$||z, r\rangle_q = D(z)S(r)|0\rangle$$
$$= \sec h^{1/2} r \exp\left[(a_q^\dagger - z^*)\tanh r\right] ||z\rangle_q \quad (35)$$

It then follows that we can prove that the states (35) and the following states:

$$_q\langle z, r| = {_q\langle z|} \sec h^{1/2} r \exp\left[(A_q - z)\tanh r\right] \quad (36)$$

make up the following completeness relation:

$$\int \frac{dz}{\pi} ||z, r\rangle_q {_q\langle z, r|} = \sec hr \int \frac{dz}{\pi} \vdots \exp\left[-|z|^2 + za_q^\dagger + z^* A_q\right.$$
$$\left. + (a_q^\dagger - z^*)\tanh r + (A_q - z)\tanh r - a_q^\dagger A_q\right]\vdots \quad (37)$$
$$= 1$$

where we have used Eq. (16).

3. QUADRATURE SQUEEZING

Here we study quadrature squeezing for the q-analogues of squeezed states $|r, q\rangle_1$ and $|r, q\rangle_2$. Now let us define the quadrature operators X_1 and X_2 as follows:

$$X_1 = \frac{a_q + a_q^\dagger}{2}, \quad X_2 = \frac{a_q - a_q^\dagger}{2i} \quad (38)$$

such that $[X_1, X_2] = i/2$, and they yield the unceratainty relation

$$(\Delta X_1)^2 (\Delta X_2)^2 \geq 1/16 \quad (39)$$

if any of the following conditions holds

$$(\Delta X_j)^2 \leq 1/4 \quad (j = 1, 2) \quad (40)$$

the field is said to be squeezed.

In order to examine the squeezing depth of the light field, we define the squeezing degree in the following form:

$$D_1(1) = 2\langle a_q^\dagger a_q\rangle + \langle a_q^{\dagger 2} + a_q^2\rangle - \langle a_q^\dagger + a_q\rangle^2 < 0 \tag{41}$$

$$D_2(1) = 2\langle a_q^\dagger a_q\rangle - \langle a_q^{\dagger 2} + a_q^2\rangle + \langle a_q^\dagger - a_q\rangle^2 < 0 \tag{42}$$

If the squeezing degree $D_j(1)$ $(j = 1, 2)$ satisfies the conditions $-1 \leq D_j(1) < 0$, it means that the field has the quadrature squeezing effect in the direction X_j $(j = 1, 2)$, and the maximum squeezing (100%) is obtained while for $D_j(1) = -1$.

By virtue of the q-analogues of squeezed states from Eqs. (26–30), we have the following expectation values of some operators:

$$\langle a_q\rangle_1 = \langle a_q^\dagger\rangle_1 = 0 \tag{43}$$

$$\langle a_q^\dagger a_q\rangle_1 = C_1^2 \sum_{n=0}^{\infty} \left(\frac{1}{2}\tanh r\right)^{2n} \frac{[2n]![2n]}{(n!)^2} \tag{44}$$

$$\langle a_q^2\rangle_1 = \langle a_q^{\dagger 2}\rangle_1 = C_1^2 \sum_{n=0}^{\infty} \left(-\frac{1}{2}\tanh r\right)^{2n+1} \times \frac{[2n+2]!}{(n!)^2(n+1)} \tag{45}$$

Obviously, under the states $|r, q\rangle_2$ the expectation values of the above operators are easily obtained via Eqs. (27), (29) and (43)–(45). By virtue of Eqs. (41)–(45) and the numerical computation method, the variations of the squeezing functions $D_j(1)$ $(j = 1, 2)$ with respect to the squeezed parameter r for fixed

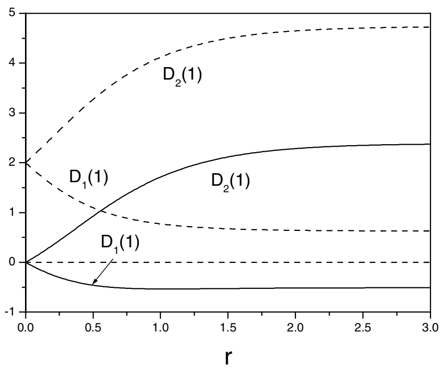

Fig. 1. Quadrature squeezing of the state $|r, q\rangle_2$ (Solid curve) and $|r, q\rangle_1$ (broken curve) for $q = 0.6$.

$q = 0.6$ are shown in detail in Fig. 1. From Fig. 1 it is clear that for the states $|\lambda, q\rangle_1$ the squeezing degree $D_1(1)$ is always negative and satisfies the squeezing conditions $-1 \leq D_1(1) < 0$ over a considerable range of r, so this means that the states $|\lambda, q\rangle_1$ exhibit squeezing in the X_1 direction. On the other hand, the squeezing degree of the states $|\lambda, q\rangle_1$ basically keep constant with the increasing of r.

4. AMPLITUDE-SQUARED SQUEEZING

In order to examine whether or not the q-analogues of squeezed states $|r, q\rangle_1$ and $|r, q\rangle_2$ exhibit amplitude-squared squeezing, let us consider the following Hermitian quadrature operators:

$$Y_1 = \frac{a_q^2 + a_q^{\dagger 2}}{2}, \quad Y_2 = \frac{a_q^2 - a_q^{\dagger 2}}{2i} \tag{46}$$

Then Y_1 and Y_2 yield the commutation relation

$$[Y_1, Y_2] = \frac{i}{2}\left[a_q^2, a_q^{\dagger 2}\right] = i(2N + 1) \tag{47}$$

and obey the unceratainty relation

$$(\Delta Y_1)^2(\Delta Y_2)^2 \geq |\langle N + 1/2\rangle|^2 \tag{48}$$

The field is said to be in an amplitude-squared squeezing state if

$$(\Delta Y_j)^2 \leq |\langle N + 1/2\rangle| \quad (j = 1, 2) \tag{49}$$

In order to examine the squeezing depth of the light field, we also define the squeezing degree in the following form:

$$D_1(2) = \frac{\left\langle a_q^4 + a_q^{\dagger 4}\right\rangle + 2\left\langle a_q^{\dagger 2} a_q^2\right\rangle - \left\langle a_q^{\dagger 2} + a_q^2\right\rangle^2}{\left\langle a_q^2 a_q^{\dagger 2}\right\rangle - \left\langle a_q^{\dagger 2} a_q^2\right\rangle} \tag{50}$$

$$D_2(2) = \frac{2\left\langle a_q^{\dagger 2} a_q^2\right\rangle - \left\langle a_q^4 + a_q^{\dagger 4}\right\rangle + \left\langle a_q^{\dagger 2} - a_q^2\right\rangle^2}{\left\langle a_q^2 a_q^{\dagger 2}\right\rangle - \left\langle a_q^{\dagger 2} a_q^2\right\rangle} \tag{51}$$

Similarly, if the squeezing degree $D_j(2)$ $(j = 1, 2)$ satisfies the conditions $-1 \leq D_j(2) < 0$, it means that the field has the amplitude-squared squeezing effect in the direction Y_j $(j = 1, 2)$, and the maximum squeezing (100%) is obtained while for $D_j(2) = -1$.

With the aid of the q-analogues of squeezed states from Eqs. (26–30), we also have the following expectation values of some operators in the states

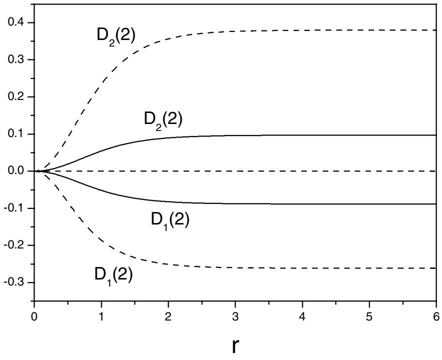

Fig. 2. Amplitude-squared squeezing of the state $|r, q\rangle_1$ (Solid curve) and $|r, q\rangle_2$ (broken curve) for $q = 0.6$.

$|\lambda, q\rangle_1$:

$$\left\langle a_q^{\dagger 2} a_q^2 \right\rangle_1 = |C_1|^2 \sum_{n=0}^{\infty} \left(\frac{1}{2} \tanh r \right)^{2n} \frac{[2n]!}{(n!)^2} [2n][2n-1] \tag{52}$$

$$\left\langle a_q^2 a_q^{\dagger 2} \right\rangle_1 = |C_1|^2 \sum_{n=0}^{\infty} \left(\frac{1}{2} \tanh r \right)^{2n} \frac{[2n+2]!}{(n!)^2} \tag{53}$$

$$\left\langle a_q^4 \right\rangle_1 = \left\langle a_q^{\dagger 4} \right\rangle_1 = |C_1|^2 \sum_{n=0}^{\infty} \left(\frac{1}{2} \tanh r \right)^{2n+2} \frac{[2n+4]!}{(n!)^2 (n+1)(n+2)} \tag{54}$$

Similarly, under the states $|r, q\rangle_2$ the expectation values of the above operators are also obtained via Eqs. (27), (29) and (52)–(54). By virtue of Eqs. (45), (50)–(54) and the numerical computation method, the variations of the squeezing degree $D_j(2)$ $(j = 1, 2)$ with respect to the squeezded parameter r for fixed $q = 0.6$ are shown in detail in Fig. 2 and From Fig. 2 it can be seen that for a wide range of r the squeezing degree $D_1(2)$ for the states $|\lambda, q\rangle_1$ and $|\lambda, q\rangle_2$ are always negative, and they always satisfy the squeezing conditions $-1 \leq D_1(2) < 0$, so it means that the states $|\lambda, q\rangle_1$ and $|\lambda, q\rangle_2$ respectively exhibit amplitude-squared squeezing in the Y_1 direction. However, the states start unsqeezed for $r = 0$ and the squeezing sets in for $r > 0$. For the states $|\lambda, q\rangle_1$ and $|\lambda, q\rangle_2$ the depth of squeezing with increasing r is rather similar for the considerable range of r. The degree of squeezing for $|\lambda, q\rangle_1$ basically have a trend to keep fixed constant with the increasing of r.

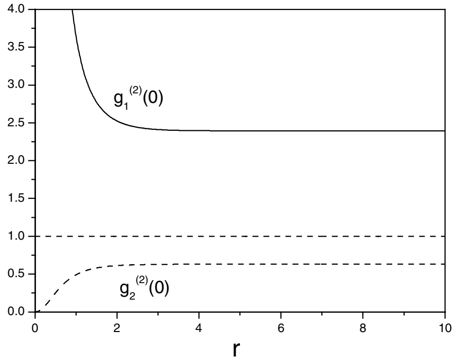

Fig. 3. Antibunching of the states $|r, q\rangle_1$ (Solid curve) and $|r, q\rangle_2$ (broken curve) for $q = 0.6$.

5. ANTIBUNCHING EFFECT

Now we shall turn to study the antibunching effect of the q-analogues of squeezed states $|r, q\rangle_1$ and $|r, q\rangle_2$. At the first step we consider the second-order correlation function $g^2(0)$, which is defined as

$$g^2(0) = \frac{\langle a_q^{\dagger 2} a_q^2 \rangle}{\langle a_q^{\dagger} a_q \rangle^2} \tag{55}$$

where $g^{(2)}(0) < 1$ means the states exhibit antibunching effect. Here by virtue of the numberical calculation resluts for $g^{(2)}(0)$, we can plot $g^{(2)}(0)$ against r for fixed $q = 0.6$ in Fig. 3. The resluts show that in the whole range of r the q-analogues of squeezed one-photon states $|r, q\rangle_2$ exhibit antibunching effect and the antibunching effect are strenthened and subsequently keep constant with increasing r, however the q-analogues of squeezed vacuum states $|r, q\rangle_1$ can not exhibit antibunching effect.

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